

Due: 2012/11/21 before class

Homework 9

Definition 1. An independent set of a hypergraph $\mathcal{H} = (V, E)$ is a set $S \subseteq V$ such that S does not contain any hyperedge, i.e. $E \cap 2^S = \emptyset$. Define $\alpha(\mathcal{H})$ to be the size of the largest independent set in \mathcal{H} .

Problem 1. What is $\alpha(\mathcal{H})$ when \mathcal{H} is the Fano configuration? (See the lecture notes for the definition of this hypergraph.)

Problem 2. Prove that, for any r and n (for simplicity, $r|n$), there is a r -uniform hypergraph \mathcal{H} with at least $\binom{n}{r}/e^r$ hyperedges, and

$$\alpha(\mathcal{H}) \geq n\left(1 - \frac{1}{r}\right).$$

Problem 3. Prove that, for any constant $c_1 > 0$, there is another constant $c_2 > 0$, such that for any 3-uniform hypergraph \mathcal{H} with n vertices and m hyperedges where $m \geq c_1 n$,

$$\alpha(\mathcal{H}) \geq \frac{c_2 n \sqrt{n}}{\sqrt{m}}.$$

Problem 4. Let A be a set of $2r + 1$ points

$$A = \{a_1, \dots, a_r, b_1, \dots, b_r, c\}.$$

Uniformly pick a random permutation σ of A . Define the random variables

$$x := |\{a_i | a_i \prec_\sigma c\}|,$$

$$y := |\{b_i | b_i \prec_\sigma c\}|.$$

Let $0 \leq p \leq 1$ be fixed.

(a) When $r = 1$ and $r = 2$, compute $\mathbb{E}_\sigma[(1+p)^x(1-p)^y]$.

(b) Prove that $\mathbb{E}_\sigma[(1+p)^x(1-p)^y] \leq 1$. (Hint: Let c_i be the number of elements in $\{a_i, b_i\}$ that are before c , (c_1, \dots, c_r) is a sequence of 0-1-2 of length r . Partition all the outcomes by conditioning on such sequences.)