Due: 2011/11/14 before class

Homework 8

Problem 1. *Consider the cycle $C_4 = (V, E)$, where $V = \{a, b, c, d\}$, and $E = \{ab, bc, cd, da\}$. We randomly color each of the 4 edges as red with probability $1/3$, and blue with probability $2/3$. For each outcome, let $R = (V, \{e \in E : e \text{ is red}\})$ and $B = (V, \{e \in E : e \text{ is blue}\})$.*
(a) The probability space contains 16 possible outcomes, for each of the resulting graph, draw it, and compute it's probability mass. (As usual, I don't care if you draw red edge with a red pen or not.)
(b) Define the random variable X to be the number of connected components in R, what is $\mathsf{E}(X)$?
(c) Define the random variable Y to be the indicator random variable for the event "B is bipartite". What is $\mathsf{E}(Y)$?
(d) Define the event $E :=$ both R and B are disconnected. What is $\mathsf{Pr}(E)$?

Problem 2. *For any positive integer n, there is a tournament on $[n]$ where the number of directed Hamilton paths (a directed path of length $n - 1$ that visits each vertex exactly once) is at least $n!/2^{n-1}$. (This is probably the first use of probabilistic method. Szele 1943.)*

Problem 3. *Let $n \geq 4$ and $t \geq 3\sqrt{n}$. Prove that, for any $n \times n$ matrix A with distinct real entries, we can transform it to a new matrix B by permuting columns, such that in B none of the rows contain any monotone sub-sequence of length t.*

In the problems below, we use some standard notations. For a graph $G = (V, E)$, and a set of colors C, a *proper (vertex) coloring* is a function $s : V \to C$ such that $s(u) \neq s(v)$ whenever uv is an edge. The *chromatic number* $\chi(G)$ is the smallest number k such that there is a proper colouring of G by $C = [k]$.

Problem 4. *Let G be a bipartite graph on n vertices, and let C be a set of $t > \log_2 n$ colors. For each vertex v, suppose we have a list of candidate colors $S(v) \subseteq C$ such that $|S(v)| > \log_2 n$. Prove that G has a proper coloring s where $s(v) \in S(v)$ for each vertex v.*

Problem 5. *Define the graph G_2 to be a simple edge. And when G_k is defined, we define G_{k+1} as follows. Suppose $V(G_k) = \{a_1, a_2, \cdots, a_t\}$, we add $t+1$ new points,*

$$V(G_{k+1}) = V(G_k) \cup \{b_1, b_2, ..., b_t, x\}.$$

And for the edges, we keep all the edges in G_k; for each b_i, connect it with all the neighbors of a_i in G_k; and finally connect x to every b_i.
(a) Draw the graphs G_3 and G_4.
(b) How many vertices are there in G_k?
(c) Show that G_k is triangle-free for any k.
(d) Suppose G_{k+1} has a proper colouring $s : V(G_{k+1}) \to [k]$ and $s(x) = k$. Show that we can modify s to another colouring s' where s' is proper on the $\{a_i\}$'s using only $[k-1]$. (This means G_k is $k-1$ colourable, and by induction, we can prove $\chi(G_k) = k$ for all k.)