Due: 2012/11/07 before class

Homework 7

Problem 1. *For each n, define the set of points in the plane*

$$A_n = \{(x, y) : x, y \in \mathbf{Z}, 1 \leq x, y \leq n\}.$$

Prove that, for any integer $c > 0$, there exists $N = N(c)$, such that no matter how we c-color A_N, there are 4 points in A_N with the same color and form an axis-parallel rectangle.

Problem 2. *For any positive integers c and r, there exists $N = N(c, r)$, such that no matter how we c-color $[N]$, there are $r + 1$ distinct positive numbers x, y_1, y_2, ..., y_r such that all the 2^r sums*

$$x + \sum_{i \in S} y_i : S \subseteq [r]$$

are not bigger than N, and have the same color. (This lemma is perhaps the first Ramsey type result in the hisotry, due to David Hilbert, 1892.)

Note. The problem above follows easily from van der Waerden theorem (why?). But try to prove it without v.d.W.

Problem 3. *Prove the stronger version of Schur's theorem: For any positive integer c, there exists $S^*(c)$ such that no matter how we color $[S^*(c)]$ by c colors, there are* distinct *$x, y, z \in [S(c)]$ of the same color such that $x+y = z$. (Hint: Prove that $S^*(c) \leq N_{2c}(3; 2)$.)*

Problem 4. *(V Chvátal)*
(a) Prove that, for any n, one can color the edges of the cube Q_n with Y and B such that there are no monochromatic copies of Q_2 (or call it C_4 if you like).
(b) Let T_1 and T_2 be two trees, discuss when it is true that, no matter how we color the edges of T_1 with Y and B, there is always a monochromatic copy of T_2.

To be more precise, describe a simple (polynomial time) algorithm, given T_1 and T_2, decide if the above property holds. Prove the correctness of your algorithm.