Do not need to hand in.

Homework 6

Problem 1. *Prove that, if G is a k-regular ($k \geq 1$) bipartite graph with bipartition A and B, then $|A| = |B|$.*

Problem 2. *For each $n > 5$, find a graph G of order n for which $|Aut(G)| = 1$. (In addition, you may check that any graph G on $[2]$, $[3]$, or $[4]$ has $|Aut(G)| > 1$.)*

Problem 3. *A proper colouring of a graph G with c colours is a mapping $f : V(G) \to [c]$ such that no adjacent vertices receive the same colour. i.e. $f(u) \neq f(v)$ whenever $u \sim v$.*
(a) Let T be a tree on $[n]$ where $n > 1$. How many proper colourings for T are there with c colours? Find a closed formula. (a.1) Give a combinatorial proof to your answer. (a.2) Give a proof using I-E. (Please do this, it is a nice exercise on I-E, combinatorial numbers, and the trees.)
(b) How many proper colourings with c colours are there for the cycle C_n? Find a closed formula and prove it using I-E.

Problem 4. *Prove that, for any $f : E(K_6) \to \{Y, B\}$, there are at least two monochromatic triangles.*

Problem 5. *Given two graphs G and H, where $V(G)$ and $V(H)$ are disjoint, their* join *is defined to be the graph with a copy of G, a copy of H and additional edges joining all the pairs between G and H. Formally,*

$$G \vee H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}).$$

Clearly, $K_3 \vee C_5$ does not contain a copy of K_6. Prove that, if its edges are coloured Y and B, there is always a monochromatic K_3.