

Do not need to hand in

## Homework 14

**Problem 1.** Let  $A$  be the adjacency matrix of the complete bipartite graph  $K_{3,3}$ , compute the eigenvalues of  $A$ .

*Solution.* One form of the adjacency matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

It has rank 2, so the null space has dimension 4, therefore  $A$  has eigenvalue 0 with multiplicity 4. For the other two eigenvalues, notice that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

So  $A$  has eigenvalues  $3, 0, 0, 0, 0, -3$ . □

**Problem 2.** Let  $n$  be a positive integer. Consider

$$S = \{(x, y, z) : x, y, z \in [n] \cup \{0\}, x + y + z > 0\}$$

as a set of  $(n+1)^3 - 1$  points in  $\mathbf{R}^3$ . We proved that  $3n$  planes are needed to touch all the points if none of the planes is allowed to touch  $(0, 0, 0)$ . Shorten the proof using Combinatorial Nullstellensatz.

*Proof.* Suppose we have  $k$  planes cover  $S$  but not the origin, the  $i$ -th plane being  $a_i x + b_i y + c_i z + d_i = 0$ . Define

$$P(x, y, z) := \prod_{1 \leq i \leq n} (x - i)(y - i)(z - i)/C - \prod_{1 \leq i \leq k} (a_i x + b_i y + c_i z + d_i),$$

where

$$C = -\frac{\prod_{1 \leq i \leq n} i^3}{\prod_{1 \leq i \leq k} d_i}.$$

So  $P$  vanishes on the set  $A^3$ , where  $A = \{0, 1, \dots, n\}$ .

If  $k < 3n$ , the degree of  $P$  is  $3n$ , and the coefficient of  $x^n y^n z^n$  is  $1/C \neq 0$ . This contradicts the Combinatorial Nullstellensatz with  $S_1 = S_2 = S_3 = A$  and  $t_1 = t_2 = t_3 = n$ .  $\square$

**Problem 3.** Let  $A_1, A_2, \dots, A_m$  be  $m$  distinct subsets of  $[n]$  such that each  $|A_i|$  is even and for any  $i \neq j$ ,  $|A_i \cap A_j|$  is odd. How big can  $m$  be? Prove your answer. (i.e., Prove the bound, and show examples where the bound is reached.)

**Lemma 1.** The matrix  $J_n - I_n$  over the binary field has rank  $n$  if  $n$  is even, and  $n - 1$  if  $n$  is odd.

*Proof.* In the binary field, plus and minus are the same, so the determinant (by its  $n!$  terms expansion) equals the number of derangements. It is 1 iff  $n$  is even. When  $n$  is odd, the rank is less than  $n$ , but the first  $n - 1$  rows and columns has full rank, so its ranks is  $n - 1$ .  $\square$

*Solution.* For  $n$  odd, the set  $\binom{[n]}{n-1}$  satisfies the requirement and has size  $n$ . For  $n$  even, the set  $\binom{[n-1]}{n-2}$  satisfies the requirement and has size  $n - 1$ . We prove this is best possible.

As in class, take the incidence matrix  $M$  where  $M_{i,j} = 1$  if  $A_i$  contains  $j$ . As matrices over the binary field,  $MM^T = J_m - I_m$ .

If  $n$  is odd, there cannot be  $n + 1$  such sets. Otherwise, the  $(n + 1) \times (n + 1)$  matrix  $MM^T = J - I$  has full rank according to the lemma. But  $M$  has only  $n$  columns. A contradiction.

If  $n$  is even, there cannot be  $n$  such sets. Otherwise, the  $n \times n$  matrix  $J - I$  has full rank, but the columns vectors of  $M$  add up to the all 0 vector, so  $M$  has rank less than  $n$ .  $\square$

**Problem 4.** Prove that, for any  $n$ , there is a two-distance set of size at least  $\binom{n+1}{2}$  in  $\mathbf{R}^n$ .

*Solution.* Let  $e_i$  be the point in  $\mathbf{R}^{n+1}$  where all the coordinates are 0, except there is a 1 on the  $i$ -th coordinate. For any  $i < j$ ,  $e_{ij} := e_i + e_j$ . There are  $\binom{n+1}{2}$  such  $e_{ij}$ 's. The distances among them are  $\sqrt{2}$  and 2. Furthermore, all those points satisfy  $x_1 + x_2 + \dots + x_n = 2$ . So they lie in the hyperplane which is isomorphic to  $\mathbf{R}^n$ .  $\square$

As usual, the last problem is always a little harder than usual.

**Problem 5.** (extra) Prove that, any two-distance set in  $\mathbf{R}^n$  has size at most  $\binom{n+2}{2}$ .

*Hint.* This is  $n + 1$  less than the bound we proved in class. Try to add  $n + 1$  new polynomials to the polynomials we used, and show that they are still linearly independent.

*Solution.* As in the notes, define, for each  $1 \leq i \leq m$ , a polynomial  $\mathbf{R}^n \rightarrow \mathbf{R}$

$$f_i(x) = (\|x - a_i\|^2 - \delta_1^2)(\|x - a_i\|^2 - \delta_2^2).$$

Expanding the polynomials, it is easy to see each  $f_i$  is a linear combination of the following  $(n + 1)(n + 4)/2$  polynomials

$$(\sum x_i^2)^2, (\sum x_i^2)x_j, x_i x_j, x_i, 1.$$

So are the polynomials  $g_0 := 1$ ,  $g_1 := x_1$ , ...,  $g_n := x_n$ . We have  $m \leq \binom{n+2}{2}$  if the  $f_i$ 's and  $g_i$ 's are linearly independent.

Suppose

$$P := \sum_i \lambda_i f_i = u + \sum_j \mu_j x_j,$$

we want to prove the  $\lambda$ 's are all 0 (therefore  $u$  and  $\mu$ 's are all 0, since the  $g_i$ 's are clearly independent).

Consider the degree 4 terms in  $P$ , e.g.  $x_1^4$ , it's coefficient

$$\sum_i \lambda_i = 0.$$

Consider the coefficient for  $x_k^3$  in  $P$ , we have

$$\forall k, \sum_i \lambda_i a_{i,k} = 0.$$

Define the matrix

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

and the vector  $v = (\lambda_1, \lambda_2, \dots, \lambda_m)$ . The above two equations says  $Av^T = 0$ . On the otherhand,  $f_i(a_i) = \delta_1^2 \delta_2^2$  and  $f_i(a_j) = 0$  whenever  $i \neq j$ . We have

$$P(a_i) = u + \sum_{1 \leq j \leq n} u_j a_{ij} = \lambda_i \delta_1^2 \delta_2^2.$$

In the matrix form, let  $w = (u, \mu_1, \dots, \mu_n)$ , we have  $wA = \delta_1^2 \delta_2^2 v$ . So,

$$w0 = wAv^T = \delta_1^2 \delta_2^2 vv^T = \delta_1^2 \delta_2^2 \sum_i \lambda_i^2,$$

which implies that the  $\lambda_i$ 's are all 0. □