

Homework 13

Problem 1. Let $f : \mathbf{R}^2 \rightarrow [3]$ be a 3-colouring of the plane. Prove that there will be two points a and b , such that $|ab| = 1$, and $f(a) = f(b)$.

Solution. First consider any equilateral triangle abc , if two of the three vertices are coloured the same, then we are done.

Otherwise, consider any pair of points c and c' with $|cc'| = \sqrt{3}$. There are points a and b such that both abc and abc' are equilateral triangles. So we are done, or any such c and c' has the same colour. Which implies that, centered at any point c , all points on the circle with radius $\sqrt{3}$ has the same color. There are certainly infinitely many pairs with distance 1 on the circle. \square

Problem 2. Let $2 \leq k \leq \sqrt{n}$. Consider a set S of n points in the plane. A line is called good if it contains at least k points from S . Prove that there are at most cn^2/k^3 good lines, where c is a constant independent of n and k .

Solution. For any good line with points appear on it as v_1, v_2, \dots, v_s in that order, where $s \geq k$. We draw $s - 1$ straight edges $v_i v_{i+1}$ ($1 \leq i < t$). This, together with S as the set of vertices, form a particular drawing of a graph G on n vertices and $m \geq t(k - 1)$ edges, where t is the number of good lines. Since the drawings of the edges lie on t straight lines, so there are at most $\binom{t}{2}$ intersections of the lines. So

$$cr(G) \leq \binom{t}{2} \quad (*)$$

On the other hand, by the crossing lemma, either $m < 4n$, which implies

$$t(k - 1) < 4n \Rightarrow t < \frac{4n}{k - 1},$$

or the crossing number is at least $\frac{m^3}{64n^2}$, which implies

$$cr(G) \geq \frac{t^3(k - 1)^3}{64n^2}$$

and by (*),

$$t \leq \frac{32n^2}{(k-1)^3}.$$

In any case,

$$t \leq \min\left(\frac{4n}{k-1}, \frac{32n^2}{(k-1)^3}\right).$$

It is an easy exercise to find a constant c such that $t \leq cn^2/k^3$. □

Problem 3. *The L_1 distance for two points in the plane is*

$$d((x, y), (x', y')) = |x - x'| + |y - y'|.$$

Let $n \geq 2$ and V be the set of points in the plane

$$V = \{(1, 3), (2, 4), (3, 1), (4, 2)\} \cup \{(n+i, 2n-i) : i = 0, 1, 2, \dots, n\}.$$

And let the hypergraph \mathcal{F} be the one with the edges

$$\{\{a, b, c\} \subseteq V : |\{a, b, c\}| = 3, d(a, b) + d(b, c) = d(a, c)\}.$$

How many different lines are there in this system?

Solution. Let $a = (1, 3)$, $b = (2, 4)$, $c = (3, 1)$, $d = (4, 2)$, and $e_i = (n+i, 2n-i)$, and $W = \{e_i | 0 \leq i \leq n\}$.

When $n = 2$, $b = e_0$ and $d = e_2$. The lines are $\overline{ab} = \{a, b\}$, $\overline{cd} = \{c, d\}$. All of \overline{ac} , \overline{bd} , \overline{ad} , \overline{bc} contains the two generating points and e_1 . $\overline{ae_1} = V - b$, $\overline{be_1} = V - a$, $\overline{ce_1} = V - d$, and $\overline{de_1} = V - c$. All the 10 lines are different.

When $n = 3$,

$$|\overline{ac}| = |\overline{bd}| = |\overline{ad}| = |\overline{bc}| = 2;$$

$$\overline{ab} = \{a, b\} \cup W - \{e_3\};$$

$$\overline{cd} = \{c, d\} \cup W - \{e_0\};$$

$$\overline{e_i e_j} = W;$$

$$\overline{ae_0} = \overline{be_0} = \{a, b, e_0\};$$

$$\overline{ae_1} = \overline{be_1} = \{a, b, e_1\};$$

$$\overline{ae_2} = \{a, b, e_2\};$$

$$\overline{be_2} = \{a, b, e_2, e_3\};$$

$$\begin{aligned}\overline{ae_3} &= \{a, e_3\}; \\ \overline{be_3} &= \{b, e_2, e_3\};\end{aligned}$$

Similarly, there are 6 lines generated by one of $\{c, d\}$ and one of the e_i 's. There are 19 different lines.

For $n \geq 4$, we have

$$\begin{aligned}\overline{ab} &= \{a, b\} \cup W; \\ \overline{cd} &= \{c, d\} \cup W; \\ \overline{ac} &= \{a, c\}; \\ \overline{bd} &= \{b, d\}; \\ \overline{ad} &= \{a, d\}; \\ \overline{bc} &= \{b, c\}; \\ \overline{ae_i} &= \overline{be_i} = \{a, b, e_i\}, 0 \leq i \leq n; \\ \overline{ce_i} &= \overline{de_i} = \{c, d, e_i\}, 0 \leq i \leq n; \\ \overline{e_i e_j} &= W\end{aligned}$$

There are $2n + 9$ different lines. □

Problem 4. *Let \mathcal{F} be a 3-uniform hypergraph on $[n]$ where among any four points there are less than 4 hyperedges. Prove that in this system there is a line equal to $[n]$, or there are at least n different lines.*

Remark. Any pair of points generate a line, some of the lines are the same (considered as sets of points). If a, b, c, d are four distinct points and $\overline{ab} = \overline{cd}$, then there are 4 hyperedges among those 4 points. Therefore, any line is repeated at most $n - 1$ times. This already proves that the number of lines is at least $n/2$.

We note the reasoning above as

Lemma 1. *If, in \mathcal{F} , $\overline{ab} = \overline{cd}$, then $\{a, b\} \cap \{c, d\} \neq \emptyset$.*

The statement clearly holds when $n = 3$. Now we use induction to prove that, when $n \geq 4$, there are always n lines, *no matter* there is a line equals $[n]$ or not. The basis, when $n = 4$, are still easy and we omit the details.

We call the unordered pair $\{a, b\}$ the *generator* of \overline{ab} . By the lemma, any two generators share a point. Assume there are two generators $\{p, x\}$ and

$\{p, y\}$ generate the same line L . Let $W = V - \{p\}$. By induction, there are at least $n - 1$ lines

$$L'_i = \overline{a_i b_i}', i = 1, \dots, n - 1$$

in $\mathcal{F} - \{p\}$. (We use ' to indicate the lines are in the system of $\mathcal{F} - \{p\}$.) Let $L_i = \overline{a_i b_i}$ be the line in the system of \mathcal{F} . $L_i = L'_i$ or $L'_i \cup \{p\}$. The L_i 's intersect W with different sets, so they are $n - 1$ different lines. If $\overline{px} \neq L_i$ for all i , then we are done by induction. If $\overline{px} = \overline{py} = \overline{a_i b_i}$ but $\{a_i, b_i\} \neq \{x, y\}$, then we have a contradiction against the lemma.

Now, suppose there are t different lines. We colour K_n with t colours, mark ab with colour i if \overline{ab} is the i -th line. By the discussion above, K_n is partitioned into t triangles and edges (*). It is easy to see that, when $n > 6$, $t \geq \binom{n}{2}/3 \geq n$. It is also easy to see (exercise again) when $n = 5, 6$, we need at least n colours in order to do (*). \square