Due: 2012/11/28 before class

Homework 10

Note: When you want to apply the L.L.L.. Specify clearly what is the probability space; what are the events; how many events do you have; what is the dependency graph you want to use and why it satisfies the definition of a dependency graph; and show your calculations about the probabilities and degrees.

Problem 1. *Given a graph $G = (V, E)$, we uniformly randomly pick σ, a permutation of V. For each vertex $v \in V$, call v a seed if $v \prec_\sigma u$ for any $uv \in E$; and define X_v to be the indicator random variable for the event that v is a seed. And define $X = \sum_{v \in V} X_v$ to be the number of seeds in an outcome.*
(a) If G is the graph on 6 vertices that consists of two vertex-disjoint triangles, what is $\mathsf{E}(X)$, and what is the probability of $X = \mathsf{E}(X)$?
(b) If G is the cycle C_6, what is $\mathsf{E}(X)$, and what is the probability of $X = \mathsf{E}(X)$?

Problem 2. *For any $n > 2$, show an example of a probability space and n events where any $n - 1$ of them are mutually independent, but the n events are not mutually independent.*

Problem 3. *Fix some integer $d \geq 1$. Let $G = (V, E)$ be a graph and for each v we have a set of candidate colours $S(v)$ where $|S(v)| = 10d$. Suppose that for each $v \in V$ and each candidate $c \in S(v)$, there are at most d vertices u such that $uv \in E$ and $c \in S(u)$. Prove that G has a proper colouring where each vertex v is coloured by one of its candidates in $S(v)$. (Hint: Finding the right collection of events will bring a short solution.)*

Problem 4. *Let $k \geq 10$ and $n = \lfloor 2^k/(10k) \rfloor$. A subset $X \subseteq [n]$ is called a* Za *if the elements of X form an arithmetic progression of length k.*
(a). *Show that for any Za, there are less than $1.25kn$ other Za's share some common points with it.*
(b). *Prove that we can color each element of $[n]$ with yellow and blue such that no Za is monochromatic.*