Version Fall 2011

Combinatorics and Graph Theory

Due to the laziness of the sincerely yours, the following notes is sketchy and lacks a lot of nice features. Especially, there are no Chinese characters in the notes, and there are almost no pictures provided. Note that Chinese is a beautiful language and in some sense the only language the instructor knows. And every mathematician should be able to draw a lot of beautiful pictures. Please understand that anything the notes lacks will be provided in abundance in the classroom.

1 Basic Counting

Example 1.1. *Counting the lattice paths.*

From $(0,0)$ to $(4,3)$, with D.P., with combinatorial explanation. From $(0,0,0)$ to (r_1, r_2, r_3).
$\binom{n}{r}$, the number of ways to pick r persons from n persons, the number of ways to pick a r element set from $\{1, 2, \cdots, n\}$.
Notation $[n] := \{1, 2, \cdots, n\}$. Notation $\binom{[n]}{r}$ as a set (of all r-element subsets of $[n]$) is defined as $\{S \subseteq [n] : |S| = r\}$. And $\binom{n}{r} = |\binom{[n]}{r}|$.
Calculating $\binom{n}{r}$. 1) Directly. 2) Recurrence and Pascal triangle (and lattice path). Notations $n!$, $(n)_k$. $\binom{n}{r} = (n)_r/r! = \frac{n!}{r!(n-r)!}$.

Exercise 1.1. *The number of mappings from $[n]$ to $[k]$ is k^n, among them $(k)_n$ are injections.*

Fact 1.1. *If p is prime and $0 < i < p$, then $\binom{p}{i} = 0$ (mod p).*

Example 1.2. *Prove that $2^p - 2 \equiv 0$ (mod p) for prime number p.*

Theorem 1.2 (Lucas 1878)**.** *Let p be a prime, $m, n \in \mathbf{N}$, write the numbers m and n in base p: $m = (m_k m_{k-1} ... m_0)$ and $n = (n_k n_{k-1} ... n_0)$; then*

$$\binom{m}{n} \equiv \prod_0^k \binom{m_i}{n_i}.$$

Equivalently,

$$\binom{m}{n} \equiv \binom{m \mod p}{n \mod p} \binom{\lfloor m/p \rfloor}{\lfloor n/p \rfloor}.$$

Proof. Let $n = tp + r$. Let $m = sp + r'$. Write out

$$\begin{aligned}\binom{m}{n} &= \frac{m(m-1)\cdots(m-r+1)}{n(n-1)\cdots(n-r+1)}\frac{(m-r)(m-r-1)\cdots(m-n+1)}{tp(tp-1)\cdots 1}\\ &\equiv \frac{m(m-1)\cdots(m-r+1)}{r!}\frac{A^t p^t s(s-1)\cdots(s-t+1)}{A^t p^t t!}\\ &\equiv \binom{r'}{r}\binom{s}{t}.\end{aligned}$$

Where $A = \prod_{i=1}^{p-1} 1$. In the last step, the first term is 0 if $r' < r$, but in any case the equation holds. □

Proof. (one more) Consider the polynomial in x over the field $\mathbf{Z}_p$: $P(x) := (1+x)^m$. $\mathcal{C}_{x^n} P = \binom{m}{n}$ (in $\mathbf{Z}_p$).
Note that $(1+x)^p = 1 + x^p$. So

$$P(x) = \prod_{i=0}^{k}(1+x)^{m_i p^i} = \prod_{i=0}^{k}(1+x^{p^i})^{m_i} = \prod_{i=0}^{k}\sum_{j=0}^{m_i}\binom{m_i}{j}x^{j\cdot p^i} = \prod_{i=0}^{k}\sum_{j=0}^{p-1}\binom{m_i}{j}x^{j\cdot p^i}.$$

Note that in the last form, there is only one way to get x^n. □

Use Lucas' theorem, consider $\mathbf{Z}_2$, we solve the following.

Example 1.3. *Find the number of odd entries in the n-th row of the Pascal triangle.*

Example 1.4. $\binom{n}{r} = \binom{n}{n-r}$. *Proof from the formula and combinatorial proof.*

More symmetric notation. Let $n = r_1 + r_2 + \cdots + r_t$. The multinomial number.

$$\binom{n}{r_1, r_2, \cdots, r_t} = \binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{n-r_1-\ldots-r_{k-1}}{r_k} = \frac{n!}{r_1!r_2!\cdots r_k!}$$

Consider the expansion of $(x+y+z)^n$, for any $r_1+r_2+r_3 = n$, the cooefficient of the term $x^{r_1}y^{r_2}z^{r_3}$ is $\binom{n}{r_1,r_2,r_3}$.

Example 1.5. $\Sigma_{r=0}^{n}\binom{n}{r} = 2^n$.

Solution. Analytic: Consider the expansion of $(1+x)^n$ and use $x = 1$.
Combinatorial: Both sides are counting the number of subsets of $[n]$. □

Example 1.6. $\Sigma_{r=0}^{n}(-1)^r\binom{n}{r} = 0$.

Solution. Analytic: Use $x = -1$ in $(1 + x)^n$.
Combinatorial: Define a mapping $f : 2^{[n]} \to 2^{[n]}$ as follows.

$$f(S) = \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S + \{1\} & \text{if } 1 \notin S \end{cases}$$

Easy to check f is its own inverse, and it is a bijection between even-subsets and odd-subsets.
Another proof: As a student pointed out, use the Pascal triangle, both even subsets and odd subsets equals the sum of the previous row. This actually gives a more detailed picture for the combinatorial proof above. (Think about it.) □

Example 1.7. *Give a combinatorial proof for* $\Sigma_{r=0}^{n}\binom{n}{r}^2 = \binom{2n}{n}$.

Example 1.8. *Give a combinatorial proof for*

$$\sum_{k=0}^{n}\binom{k}{a}\binom{n-k}{b} = \binom{n+1}{a+b+1}.$$

Solution. L.h.s: partition all the choices by where is the $(a + 1)$-st seleted person. □

Example 1.9. *Give a combinatorial proof for*

$$\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Solution. Pick n person from n girls and n boys, one of them is the captain who must be a girl. □

Example 1.10.

$$\Sigma_{r=0}^{n} r\binom{n}{r} = n2^{n-1}.$$

Algebraic: Take dirivative of $(1 + x)^n$ at $x = 1$, or direct algebraic manipulation.
Combinatorial: Both sides counting the sum of the size of all subsets.
This kind of double counting is usually a nice trick.

Example 1.11. *Let $f(n)$ be the number of divisors of n, what is $f(n)$ and what is the average of the first N numbers $\frac{1}{n}\sum_{n\leq N} f(n)$?*

Example 1.12. *Number of ways to put n* identical *balls into r labeled boxes. Formally, the number of solutions to*

$$x_1 + x_2 + \cdots + x_r = n$$

, where $x_i \in \mathbf{N}_0$

Answer. $\binom{n+r-1}{r-1}$, combinatorial explanation.

Example 1.13. *The number of solutions to*

$$x_1 + x_2 + \cdots + x_r = n$$

, where $x_i \in \mathbf{N}$

Answer. $\binom{n-1}{r-1}$, combinatorial explanation.

Example 1.14. *We used $x = -1$ in the expansion of $(1+x)^n$ to get $\sum \binom{n}{2k}$, now use the roots of $z^3 = 1$ to compute*

$$\sum_{k=0}^{\lfloor n/3 \rfloor} \binom{n}{3k}.$$

Check that when $n = 9$ the above equals $(2^9 - 2)/3$.

2 Catalan Numnbers

A Vašek style quotation: They are called Catalan numbers because they are not first discovered by Catalan.
Here is the problem from a letter from Euler to Goldbach in 1751.

Example 2.1. *In how many ways can one triangulate a labeled $(n+2)$-gon by $n-1$ diagonals?*

Let the answer be C_n, we have $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, and we will be glad to define $C_0 = 1$.

Exercise 2.1.

$$C_n = \sum_{0}^{n-1} C_i C_{n-1-i}$$

Definition 2.1. *The Catalan number* $C(n) := \frac{1}{n+1}\binom{2n}{n}$.

Theorem 2.1 (Euler 1751)**.** *The number of ways to triangulate an* $(n+2)$*-gon by* n *diagonals is* $C(n)$.

Proof. Let $G(x)$ be the g.f. for $\{C_n\}$, expand G^2 we get $G^2(x) = (G(x)-1)/x$. So,

$$G(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

One way or another we will pick the minus here.

$$2xG(x) = 1 - (1-4x)^{1/2}$$

and we get the answer by using Taylor. $\square$

One may already know many solutions as well many problems that leads to the same answer. We briefly mention just a few.

Proof. (one more for the triangulation problem) We establish a mapping as follows.
For a triangulation of the $(n+2)$-gon $a_1a_2...a_{n+2}$, pick any of its edge or diagonal and give it a direction ($4n+2$ choices).
For a triangulation of the $(n+3)$-gon $a_1a_2...a_{n+3}$, pick any of its edge except a_1a_2 ($n+2$ choices).
For any animal of the first kind, we duplicate the picked edge and open the two copies from the tail (of the picked direction). For any one of the second kind, we collapse the picked edge e, assume e was in the triangle efg, then f and g now become merged into one. It is your job to mediate and see this is a one-one mapping between $C_n \times (4n+2)$ and $C_{n+1} \times (n+2)$. $\square$

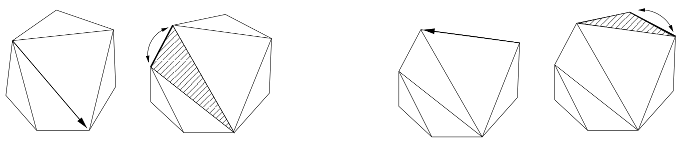

Figure 1: Some examples of the bijection.

Example 2.2. *There are* $\binom{2n}{n}$ *lattie paths from* $(0, 0)$ *to* (n, n)*. How many of them never go below the diagonal* $y = x$*?*

It is a little more convenient to turn our pictures 45 degrees (by viewing up and right $\to +1$ and -1). Define a walk from $(0, 0)$ where each step takes either $(1, 1)$ or $(1, -1)$. Restate the problem

Example 2.3. *There are* $\binom{2n}{n}$ *walks from* $(0, 0)$ *to* $(2n, 0)$*. How many of them never go below the* x*-axis?*

Solution. We count the number of "bad" walks, i.e., those go below the x axis at least once. Focus on the first time it does this, it reached the line $y = -1$, we take the walk from that point to the end, reflect it about the line $y = -1$. It becomes a walk ends at $(2n, -2)$.
It is (your time to mediate again) easy to see that this gives a bijection between the bad walks and all the walks ends at $(2n, -2)$. So the number of bad walks is $\binom{2n}{n-1}$. $\square$