Combinatorics and graph theory, Midterm

1 Section I

Problem 0. What is your name? ________________

For each of the following, fill in the blanks with the decimal representation of an integer.

Problem 1. *(5 points) There are* ________ *ordered triples* (A, B, C) *such that* $A \subseteq B \subseteq C \subseteq [5]$.

Problem 2. *(5 points) Define the set* $A = \{x : x \in [2011], 3|\binom{2011}{x}\}$*. The size of* A *is* ________.

Problem 3. *(5 points) A matrix is* good *if in each row there are no repeated elements, and in each column there are no repeated elements. There are* ________ *good matrices with* 2 *rows and* 6 *columns whose elements are all from* [6].

Problem 4. *(5 points) In the* 5*-dimensional cube* Q_5*, the length of any shortest path from the all* 0 *vector to the all* 1 *vector is* ________. *There are* ________ *such paths.*

Problem 5. *(5 points) Construct a digraph on* [2000] *where there is an edge from* i *to* j *if* $i|j$ *and* j/i *is a prime number. In this graph, there are* ________ *paths from* 1 *to* 210*; there are* ________ *paths from* 2 *to* 1024*; and there are* ________ *paths from* 1 *to* 2000.

Problem 6. *(5 points) Let* G *be the graph in Figure1, the size of* $Aut(G)$ *is* ________.

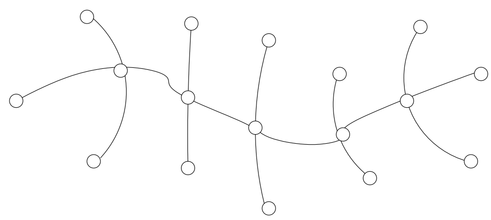

Figure 1: a graph

Problem 7. *(5 points) There are* ________ *ways to partition* [12] *into* 4 *parts such that each part has* 3 *elements.*

2 Section II

Solve 5 of the following 6 problems.

Problem 8. *(13 points) A digraph is called* balanced *if for each vertex v, $|d^+(v) - d^-(v)| \le 1$. Prove that, every simple graph G has a balanced orientation.*

Solution. (Due to Qu Jun and Liu Sizhuang) Clearly we may assume the graph is connected. Because the number of vertices with odd degree is even, say $2t$, we arbitarily pair them, and add one (magic) edge to each pair. Thus get a graph G' which is Eulerian, orient the edges according to a Eulerian path, so every vertex has the same in-degree and out-degree. Now remove the magic edges to get an orientation of G, each vertex has at most one edge removed. □

Problem 9. *(13 points) Given $n \ge 1$, we construct a graph on the set of all permutations of $[2n]$. Two permutations π and π' are adjacent if (1) π and π' are the same except on two positions i and $i+1$; and (2) $|\pi(i) - \pi(i+1)| = n$. Explain that the number of connected components in this graph is $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(2n-k)!$.*

In order to use PIE, you need to write clearly what is the set you are couting, and what are the elements you want to exclude.

Solution. For each π, consider all the pairs $(i, i+n)$, call it a loose pair if i and $i+n$ occupy adjacent positions in π. By the definition of this graph, in each step one can switch the order inside a loose pair, but all the loose pairs stay the same. π and π' are reachable from each other if they have the same loose pairs on the same positions. Suppose π has k loose pairs, its component has exactly 2^k points. We can define a representitive for each connected component, π is a representative iff in any loose pair, i always come before $i+n$. We just want to count the number of permutations where i is before $i+n$ whenever they are next to each other.
Let E_i be the set of permutations where $n+i$ comes right before i. And S be the set of all the permutations. It is easy to see, when $|M| = k$, $N(M) := |\cap_{i\in M} E_i| = (2n-k)!$. (The k pairs are grouped together, with the order inside each group fixed, and all the others are free to permute.)

Thus $N(k) = \sum_{|M|=k} N(M) = \binom{n}{k}(2n-k)!$. So, by PIE, the number of good elements we want to count is

$$|S - \cup E_i| = \sum_{k=1}^{n} N(k) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} (2n-k)!.$$

$\square$

Problem 10. *(13 points) Suppose G is a graph on n vertices and $G \cong \overline{G}$. Prove that (a) G is connected; (b) Either n or $n-1$ is a multiple of 4; (c) If $n = 4k+1$ for some integer k, then there is a vertex v such that $deg(v) = 2k$.*

Solution. (a). $G \cong \overline{G}$, so both of them are connected, or both of them are disconnected. But in class we proved that the second case cannot happen.
(b). $G \cong \overline{G}$, so they have the same number of edges m, and the total number of edges in G and $\overline{G}$ is just $\binom{n}{2}$, so $\binom{n}{2} = 2m$ is an even number. It is easy to see n mod 4 is 0 or 1.
(c). Define $A = \{v \in V : d_G(v) < 2k\}$ and $B = \{v \in V : d_G(v) > 2k\} = \{v \in V : d_{\overline{G}}(v) < 2k\}$. Let π be an isomorphism from G to $\overline{G}$. Since π preserves degrees, it mapped A into B and B into A, so $|A| \leq |B|$ and $|B| \leq |A|$, i.e. $|A| = |B|$. Since the total number of points is odd, there exists v not in A nor B, i.e. $d(v) = 2k$. $\square$

Problem 11. *(13 points) Let T be the Stirling number of the 1st kind $T = s(100, 50)$. Compute $T \mod 3$. Justify your answer.*

Solution. We know the polynomials

$$\sum_{k=1}^{100} s(100,k)x^k = x(x-1)(x-2)...(x-99).$$

So $s(100, 50)$ mod 3 is the same as

$$\mathcal{C}_{50}\left[x^{34}(x-1)^{33}(x+1)^{33}\right] = \mathcal{C}_{16}[(x^2-1)^{33}] = \binom{33}{8}.$$

The last one is 0 (mod 3) by Lucas' theorem. $\square$

Problem 12. *(13 points) Prove that, for each positive integer c, there is a number $N = N(c)$, no matter how we color* all *the subsets of $[N]$ with c colors, one can always find two* nonempty disjoint *subsets A and B such that A, B, and $A \cup B$ have the same color.*

Solution. We claim $N(c) = N_c(3; 2)$ is enough. Suppose $N = N_c(3; 2)$ and s is a coloring of all the subsets of $[N]$. We color the edges of the complete graph K_N as follows. For $i < j$, the edge ij is colored with $s(\{i, i+1, ..., j-1\})$ (*). By the definition of $N_s(3; 2)$, there are $i < j < k$ s.t. the color of edges ij, jk, and ik are the same. Let $A = \{i, ..., j-1\}$, $B = \{j, ..., k-1\}$, so $A \cup B = \{i, ..., k-1\}$. By (*), they have the same color by s. $\square$

Problem 13. *(13 points) Color the edges of K_n by yellow and blue. A cycle is called* monochromatic *if all its edges have the same color. Prove that, if there is a monochromatic cycle of length $2k+1$ for some $k > 2$, then there is also a monochromatic cycle of length $2k$.*

Solution. (Due to Liu Sizhuang) w.l.o.g., there is a yellow cycle $a_0 a_1 ... a_{2k}$. If there is any i such that $a_i a_{i+2}$ (here the addition is taken modulo n) is yellow, then we get a yellow cycle of length $2k$. Otherwise, all the "length 2" edges are blue, and since $(2, n) = 1$, they form a blue cycle of length $2k+1$, $a_0 a_2 ... a_{2k} a_1 a_3 ... a_{2k-1}$.
Similarly, we can either find a blue cycle of length $2k$, or all the "length 4" edges are yellow. In the latter case, $a_0 a_4 a_5 a_1 a_2 a_6 a_7 a_8 ... a_{2k}$ is a yellow cycle of length $2k$. $\square$