Due: 2011/11/16 before class

Homework 7

Problem 1 (The Chvátal graph)**.** *The graph in Figure 1 is due to Vašek Chvátal (1970). It is 4-regular and has girth 4. Prove that it is 4-colourable.*

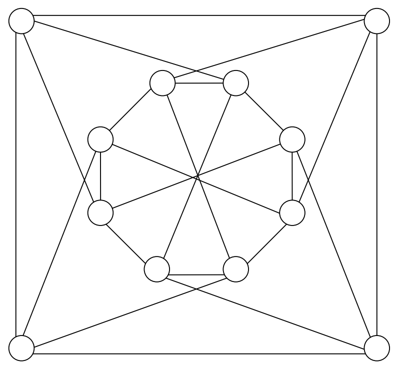

Figure 1: The Chvátal graph.

Problem 2. *Define the graph G_2 to be a simple edge. And when G_k is defined, we define G_{k+1} as follows. Suppose $V(G_k) = \{a_1, a_2, \cdots, a_t\}$, we add $t+1$ new points,*

$$V(G_{k+1}) = V(G_k) \cup \{b_1, b_2, ..., b_t, x\}.$$

And for the edges, we keep all the edges in G_k; for each b_i, connect it with all the neighbors of a_i in G_k; and finally connect x to every b_i.
(a) Draw the graphs G_3 and G_4.
(b) How many vertices are there in G_k?
(c) Show that G_k is triangle-free for any k.
(d) Suppose G_{k+1} has a proper colouring $s : V(G_{k+1}) \to [k]$ and $s(x) = k$. Show that we can modify s to another colouring s' where s' is proper on the $\{a_i\}$'s using only $[k-1]$. (This means G_k is $k-1$ colourable, and by induction, we can prove $\chi(G_k) = k$ for all k.)

Problem 3. *Let A be a set of $2r+1$ points*

$$A = \{a_1, ..., a_r, b_1, ..., b_r, c\}.$$

Uniformly pick a random permutation σ of A. Define the random variables

$$x := |\{a_i | a_i \prec_\sigma c\}|,$$

$$y := |\{b_i | b_i \prec_\sigma c\}|.$$

Let $0 \leq p \leq 1$ be fixed.
(a) When $r = 1$ and $r = 2$, compute $\mathsf{E}_\sigma[(1+p)^x(1-p)^y]$.
(b) Prove that $\mathsf{E}_\sigma[(1+p)^x(1-p)^y] \leq 1$. (Hint: Let c_i be the number of elements in $\{a_i, b_i\}$ that are before c, $(c_1, ..., c_r)$ is a sequence of 0-1-2 of length r. Partition all the outcomes by conditioning on such sequences.)