Due: 2011/11/09 before class

Homework 6

Problem 1. *Consider the cycle $C_4 = (V, E)$, where $V = \{a, b, c, d\}$, and $E = \{ab, bc, cd, da\}$. We randomly color each of the 4 edges as red with probability $1/3$, and blue with probability $2/3$. For each outcome, let $R = (V, \{e \in E : e \text{ is red}\})$ and $B = (V, \{e \in E : e \text{ is blue}\})$.*
(a) The probability space contains 16 possible outcomes, for each of the resulting graph, draw it, and compute it's probability mass. (As usual, I don't care if you draw red edge with a red pen or not.)
(b) Define the random variable X to be the number of connected components in R, what is $\mathsf{E}(X)$?
(c) Define the random variable Y to be the indicator random variable for the event "B is bipartite". What is $\mathsf{E}(Y)$?
(d) Define the event $E :=$ both R and B are disconnected. What is $\mathsf{Pr}(E)$?

Problem 2. *For any positive integer n, there is a tournament on $[n]$ where the number of directed Hamilton paths (a directed path of length $n-1$ that visits each vertex exactly once) is at least $n!/2^{n-1}$. (This is probably the first use of probabilistic method. Szele 1943.)*

Problem 3. *Prove the following by induction: Let $v_1, v_2, ..., v_n$ be n vectors in $\mathbf{R}^n$ and $|v_i| = 1$ for all i. Then there exists $\epsilon_1, ..., \epsilon_n$ such that each ϵ_i is either 1 or -1, and $|\sum_{i=1}^{n} \epsilon_i v_i| \leq \sqrt{n}$.*

Problem 4. *Prove the proposition in the previous problem using probabilistic method. (You may follow this line: Pick each ϵ_i randomly, let $v = \sum_i \epsilon_i v_i$ and define the random variable*

$$X = |v|^2 = (v, v),$$

where $(,)$ is the inner product. Expand the inner product.)

Problem 5. *For a graph $G = (V, E)$, and a set of colors C, a* proper (vertex) coloring *is a function $s : V \to C$ such that $s(u) \neq s(v)$ whenever uv is an edge.*
Let G be a bipartite graph on n vertices, and let C be a set of $t > \log_2 n$ colors. For each vertex v, suppose we have a list of candidate colors $S(v) \subseteq C$ such that $|S(v)| > \log_2 n$. Prove that G has a proper coloring s where $s(v) \in S(v)$ for each vertex v.