Due: 2011/11/09 before class

Homework 6

Problem 1. *Consider the cycle* $C_4 = (V, E)$*, where* $V = \{a, b, c, d\}$*, and* $E = \{ab, bc, cd, da\}$*. We randomly color each of the 4 edges as red with probability* $1/3$*, and blue with probability* $2/3$*. For each outcome, let* $R = (V, \{e \in E : e \text{ is red}\})$ *and* $B = (V, \{e \in E : e \text{ is blue}\})$*.*
(a) The probability space contains 16 possible outcomes, for each of the resulting graph, draw it, and compute it's probability mass. (As usual, I don't care if you draw red edge with a red pen or not.)
(b) Define the random variable X *to be the number of connected components in* R*, what is* $\mathsf{E}(X)$*?*
(c) Define the random variable Y *to be the indicator random variable for the event "*B *is bipartite". What is* $\mathsf{E}(Y)$*?*
(d) Define the event $E :=$ *both* R *and* B *are disconnected. What is* $\mathsf{Pr}(E)$*?*

Problem 2. *For any positive integer* n*, there is a tournament on* $[n]$ *where the number of directed Hamilton paths (a directed path of length* $n-1$ *that visits each vertex exactly once) is at least* $n!/2^{n-1}$*. (This is probably the first use of probabilistic method. Szele 1943.)*

Problem 3. *Prove the following by induction: Let* $v_1, v_2, ..., v_n$ *be* n *vectors in* $\mathbf{R}^n$ *and* $|v_i| = 1$ *for all* i*. Then there exists* $\epsilon_1, ..., \epsilon_n$ *such that each* ϵ_i *is either* 1 *or* -1*, and* $|\sum_{i=1}^n \epsilon_i v_i| \leq \sqrt{n}$*.*

Problem 4. *Prove the proposition in the previous problem using probabilistic method. (You may follow this line: Pick each* ϵ_i *randomly, let* $v = \sum_i \epsilon_i v_i$ *and define the random variable*

$$X = |v|^2 = (v, v),$$

where $(,)$ *is the inner product. Expand the inner product.)*

Problem 5. *For a graph* $G = (V, E)$*, and a set of colors* C*, a* proper (vertex) coloring *is a function* $s : V \to C$ *such that* $s(u) \neq s(v)$ *whenever* uv *is an edge.*
Let G *be a bipartite graph on* n *vertices, and let* C *be a set of* $t > \log_2 n$ *colors. For each vertex* v*, suppose we have a list of candidate colors* $S(v) \subseteq C$ *such that* $|S(v)| > \log_2 n$*. Prove that* G *has a proper coloring* s *where* $s(v) \in S(v)$ *for each vertex* v*.*