

Homework 5

Problem 1. Given two graphs G and H , where $V(G)$ and $V(H)$ are disjoint, their join is defined to be the graph with a copy of G , a copy of H and additional edges joining all the pairs between G and H . Formally,

$$G \vee H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}).$$

Clearly, $K_3 \vee C_5$ does not contain a copy of K_6 . Prove that, if its edges are colored Y and B , there is always a monochromatic K_3 .

Problem 2. For each n , define the set of points in the plane

$$A_n = \{(x, y) : x, y \in \mathbf{Z}, 1 \leq x, y \leq n\}.$$

Prove that, for any integer $c > 0$, there exists $N = N(c)$, such that no matter how we c -color A_N , there are 4 points in A_N with the same color and form an axis-parallel rectangle.

Problem 3. For any positive integers c and r , there exists $N = N(c, r)$, such that no matter how we c -color $[N]$, there are $r + 1$ numbers x, y_1, y_2, \dots, y_r such that all the 2^r sums

$$x + \sum_{i \in S} y_i : S \subseteq [r]$$

are not bigger than N , and have the same color. (This lemma is perhaps the first Ramsey type result in the history, due to David Hilbert, 1892.)

Problem 4. Prove the stronger version of Schur's theorem: For any positive integer c , there exists $S^*(c)$ such that no matter how we color $[S^*(c)]$ by c colors, there are distinct $x, y, z \in [S(c)]$ of the same color such that $x + y = z$. (Hint: Prove that $S^*(c) \leq N_{2c}(3; 2)$.)