Due: 2011/10/19 before class

# Homework 4

**Problem 1.** *Prove that, if $G$ is a $k$-regular ($k \geq 1$) bipartite graph with bipartition $A$ and $B$, then $|A| = |B|$.*

**Problem 2.** *For each $n$, find $n+1$ distinct subsets of $[n]$, $A_i$ ($1 \leq i \leq n+1$) such that for each $x \in [n]$, there are $1 \leq i < j \leq n+1$ where $A_i - x = A_j - x$.*

**Problem 3.** *Prove that, in a connected graph $G$, any 2 longest paths share a common vertex. (Note: write your proof as convincing as possible.)*

**Problem 4.** *Let $T$ be a tree, $T_i$ ($1 \leq i \leq k$) be subtrees of $T$. Prove that, if any two of the $T_i$'s share a common vertex, then all the $T_i$'s share a common vertex.*

**Problem 5.** *For any positive integer $n$, define a graph $G = (V, E)$, where $V$ consists of points in the plane $(i, 0)$ for $i = 0, 1, 2, ..., n+1$, $(i, 1)$ and $(i, -1)$ for $i = 1, 2, ..., n$. Two vertices are adjacent if their distance in the plane is 1. Find the size of $Aut(G)$.*

**Problem 6.** *For each $n > 5$, find a graph $G$ of order $n$ for which $|Aut(G)| = 1$.*

**Problem 7.** *Let $G$ be a graph on $n$ vertices ($n > 3$) with no vertex of degree $n-1$. Suppose that for any two vertices of $G$, there is a unique vertex adjacent to both of them.*
*(a) If $u$ and $v$ are not adjacent, prove that they have the same degree. (Hint: Construct a bijection between the two sets of neighbors.)*
*(b) Show that $G$ is $k$-regular for some $k$.*
*(c) Express $n$ in terms of $k$.*