Do not need to submit.

Homework 12

Problem 1. *Let e be an edge in K_n. What is the number of spanning trees that contain e as an edge?*

Solution. By Cayley's formula, there are n^{n-2} spanning trees in K_n. Each tree has $n-1$ edges, so there are $(n-1)n^{n-2}$ (tree, edge) containing pairs. Since each of the $\binom{n}{2}$ edges looks the same, so each edge is contained in $(n-1)n^{n-2}/\binom{n}{2} = 2n^{n-3}$ trees. □

Problem 2. *Let A be the adjacency matrix of the complete bipartite graph $K_{3,3}$, compute the eigenvalues of A.*

Solution. One form of the adjacency matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

It has rank 2, so the null space has dimension 4, therefore A has eigenvalue 0 with multiplicity 4. For the other two eigenvalues, notice that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

and

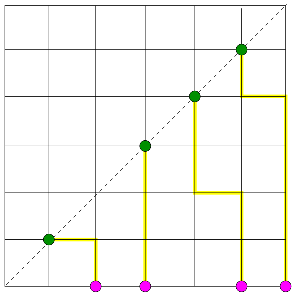
$$A \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

So A has eigenvalues $3, 0, 0, 0, 0, -3$. □

Problem 3. *Let $a_1 < ... < a_n$ and $b_1 < ... < b_n$ be two sequence of positive integers. Let M be the matrix where the (i,j) entry is $\binom{a_i}{b_j}$. Then $\det M \geq 0$, and it is 0 iff $a_i < b_i$ for some i.*

Hint. Try to use the Lindström-Gessel-Viennot lemma. Find a d.a.g. and two sets of n points where there are $\binom{a_i}{b_j}$ different paths from the i-th point in the firt set to the j-th point in the second set. Naturally we may consider the lattice paths.

More hint. The following picture shows an example when $\{a_i\} = \{2, 3, 5, 6\}$ and $\{b_i\} = \{1, 3, 4, 5\}$. One of the vertex disjoint path system is highlighted.



Proof. Let N be a number bigger than all the a_i and b_i's. Consider the following d.a.g.: The vertices are the points $\{(x, y) : 0 \leq x, y \leq N\}$. (In fact we only need the part that is below the diagonal.) The edges are all the vertical or horizontal unit length edges connecting two neighbouring vertices, and the directions are either to the right $((x, y) \to (x+1, y))$, or to the bottom $((x, y) \to (x, y-1))$.

Let A be the sequence of vertices $A = (v_1, v_2, ..., v_n)$, where $v_i = (b_i, b_i)$. Let B be the sequence of vertices $B = (u_1, u_2, ..., u_n)$, where $u_i = (a_i, 0)$. It is easy to check that the number of different paths from v_i to u_j is $\binom{a_i}{b_j}$. Now if we put weight 1 on every edge, then all the paths has weight 1, and

$$w(v_i, u_j) := \sum_{P: v_i \to u_j} w(P) = \binom{a_i}{b_j}.$$

Moreover, the weight of any path system, defined as the product of the weights of paths, is also 1. So, by Lindström-Gessel-Viennot,

$$\det M = \sum_{\mathcal{P} \in \mathcal{F}^*(A,B)} \operatorname{sign}\mathcal{P} w(\mathcal{P}) = \sum_{\mathcal{P} \in \mathcal{F}^*(A,B)} \operatorname{sign}\mathcal{P}.$$

Note that for any path system $\mathcal{P}$ to be vertex disjoint, it is easy to see (convince yourself, remember $a_1 < a_2 < ...$ and $b_1 < b_2 < ...$) we must have v_1 goes to u_1, and then v_2 goes to u_2, etc. So sign$\mathcal{P} = 1$ for all $\mathcal{P} \in \mathcal{F}^*(A, B)$. Therefore $\det M$ equals the number of *vertex disjoint* path systems from A to B, which is surely non-negative. Moreover, it is easy to see such path system exists iff $a_i \geq b_i$ for all i. $\square$

Consider the following a bonus problem. Since we did not have time to cover the relevant thoughts in class, it has nothing to do with the final exam.

Problem 4 (I.M.O. 2007)**.** *Let n be a positive integer. Consider*

$$S = \{(x, y, z) : x, y, z \in [n] \cup \{0\}, x + y + z > 0\}$$

as a set of $(n+1)^3 - 1$ points in $\mathbf{R}^3$. Determine the smallest number of planes, the union of which contains S but does not inlcude $(0, 0, 0)$.

Remark. The answer is $3n$. It is easy to see $3n$ planes are doable. Actually there are many ways to do this: (1) Take the planes $x = 1$, $x = 2$, ..., $x = n$, $y = 1$, ..., $y = n$, $z = 1$, ..., $z = n$. (2) Take the planes $x + y + z = 1$, ..., $x + y + z = 3n$. (3) Some combinations of (1) and (2).

Remark. The corresponding 2d problem, where we have a square instead of a cube, has a simple elementary proof: First we consider all the horizontal or vertical lines, then the rest un-covered points still form a 2d array of points. We can focus on the "boundary" of the array, and since no more horizontal or vertical lines, each line will intersect with the boundary at most twice.

Hint. Each plane gives a linear polynomial of the form $a_i x + b_i y + c_i z + d_i = 0$, consider the product of all these polynomials.

More hint. To keep the illustration managable, let us look at the algebraic method on the simplest case in 2d. Where we have 4 points $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. We inspect some solutions to cover the 3 points with 2 lines.
1. $x = 1$ and $y = 1$. $(x - 1)(y - 1) = xy - x - y + 1$.
2. $x + y = 1$ and $x + y = 2$. $(x + y - 1)(x + y - 2) = 2xy + x^2 - 3x + y^2 - 3y + 2$.
3. $x + y = 1$ and $y = 1$. $(x + y - 1)(y - 1) = xy - x + y^2 - 2y + 1$.
If you happen to imagine y^2 to be the same as y and x^2 to be x, then these 3 polynomials are essentially the same. This is not an accident. If $x = 0$ or $x = 1$, indeed $x^2 = x$.

You may know the fact that if a polynomial of degree at most n has $n + 1$ roots, then it must be the zero polynomial. The following multi-dimensional generalization is (should be) also well known, and can be proved inductively from the 1d case. We omit the proof.

Lemma 1. *If $P(x, y, z)$ is a polynomial in x, y, z, and each term has degree at most (n, n, n). And suppose that there are $x_1 < x_2 < ... < x_{n+1}$, $y_1 < y_2 ... < y_{n+1}$, and $z_1 < ... < z_{n+1}$ such that $P(x_i, y_j, z_k) = 0$ for all i, j, k, then P is the zero polynomial.*

Consider $P - Q$, we have

Corollary 2. *If $P(x, y, z)$ and $Q(x, y, z)$ are two polynomials with degrees at most (n, n, n), and if $P(x, y, z) = Q(x, y, z)$ on all $0 \le x, y, z \le n$, then they are the same polynomial.*

Now we prove that in our problem, we need at least $3n$ planes.

Proof. Suppose we have a solution with t planes, the i-th plane is $a_i x + b_i y + c_i z + d_i = 0$. Define

$$P(x, y, z) := \prod_{1 \le i \le t} (a_i x + b_i y + c_i z + d_i).$$

Because each point is covered by at least one of the planes, and $(0, 0, 0)$ is not. So $P(x, y, z) = 0$ for all $(x, y, z) \in S$, and $P(0, 0, 0) \neq 0$.
Consider

$$R(x) = x^{n+1} - x(x-1)(x-2)...(x-n)$$

It is a polynomial of degree less than $n+1$, and $R(x) = x^n$ for all $0 \le x \le n$. Now we start from P and do a sequence of operations. Whenver there is a term $x^a y^b z^c$ where $a > n$, we replace x^{n+1} by $R(x)$.Formally, $x^a y^b z^c \to x^{a-n+1} R(x) y^b z^c$. By doing this, the degree of x is reduced by at least 1. And importantly, let the resulting polynomial be P', we have $P(x, y, z) = P'(x, y, z)$ for all $0 \le x, y, z \le n$.
We keep doing this, and similarly for y and z. It is easy to see, after finite number of steps, we get a polynomial P^* such that
(a). $P^*(x, y, z) = P(x, y, z)$ for all $0 \le x, y, z \le 0n$.
(b). The degree of P^* is at most (n, n, n).
Now we plug in our trivial solution, let

$$Q(x, y, z) := (x-1)(x-2)...(x-n)(y-1)...(y-n)(z-1)...(z-n).$$

Q also vanishes on $0 \le x, y, z \le n$ except $(0,0,0)$, and Q also has degrees at most (n,n,n). So, after multiplying a constant, $P^* = cQ$ on $\{0, 1, ..., n\}^3$. By the lemma, $P^* \equiv cQ$ as polynomials.
Observe that the highest degree term in Q is $x^n y^n z^n$, so P^* has that term. On the other hand, if in the beginning we have $t < 3n$ planes, each terms $x^a y^b z^c$ has $a+b+c < 3n$. And clearly the reduction will not create any term that increases $a+b+c$, so $x^n y^n z^n$ will not appear in P^*, a contradiction. $\square$

Remark. The whole proof indeed follows the outline of a well known method in algebraic combinatorics. Search for *Combinatorial Nullstellensatz.*