Due: 2011/12/14 before class

Homework 11

Problem 1. *Let n be a positive integer, $r < n/2$, $A = \binom{[n]}{r}$, and $B = \binom{[n]}{r+1}$. Construct a bipartite graph G on $A + B$, where $x \in A$ is connected to $y \in B$ iff $x \subseteq y$ (i.e. the set y is x plus one element). Prove that there is a perfect matching for $|A|$ in G.*

Remark. This is similar to our proof that any regular bipartite graph has a perfect matching.

Proof. We use Hall's theorem. All we need to show is that the graph satisfies Hall's condition. i.e. for any $X \subseteq A$, $|N(X)| \geq |X|$.
Suppose X is a subset of $\binom{[n]}{r}$ where $|X| = a$ and $|N(X)| = b$. Suppose there are m edges between X and $N(X)$. For each element in X, it has $n-r$ neighbours in B. So $m = a(n-r)$ (*1). On the other hand each element in B has $(r+1)$ neighbours in A, some of them may not be in X, so $m \leq b(r+1)$ (*2). Since $r < n/2$, we have $r+1 \leq n-r$. (*1) and (*2) gives $a \leq b$. $\square$

Problem 2. *Prove that, when n is odd, there are exactly two extremal cases for Sperner's theorem.*

Proof. Let $n = 2k+1$. Let $A = \binom{[n]}{k}$ and $B = \binom{[n]}{k+1}$. The two extremal cases are: either take all A, or all B. By the proof of Sperner's theorem, we know the size of each subset must be k or $k+1$. Now we prove that we can not have $\binom{n}{k}$ subsets where both A and B have some contribution.
Prove by contradiction. Suppose there is $\emptyset \neq X \subseteq A$ and $\emptyset \neq Y \subseteq B$ form an antichain of size $T = \binom{n}{k}$. Consider the bipartite graph between A and B as in the previous problem. Let $Y' := N(X)$. Since $X \cup Y$ form an antichain, $Y \cap Y' = \emptyset$. So $|Y| + |Y'| \leq T$. Because we assumed $|Y| + |X| = T$. We show a contradiction by proving $|X| < |Y'|$.
Problem 1 tells that there is a perfect matching for A, so clearly $|X| \leq |Y'|$. The proof also tells that, for the equality to hold, Y' has no other neighbours outside X. But this means, in the bipartite graph (A, B), $X \cup Y'$ is disconnected from the rest of the world.
However, (A, B) is connected: From any subset $S \in A$, we keep adding and removing one vertex each time, and get any other set in A or B. $\square$

Problem 3. *Given n real numbers a_1, ..., a_n such that each $|a_i| \geq 1$. For any subset $A \subseteq [n]$, define $S(A) = \sum_{i \in A} a_i$. If A_1, ..., A_m are m distinct subsets of $[n]$ such that $|S(A_i) - S(A_j)| < 1$ for any $i \neq j$, how big can m be? Justify your answer. (i.e., For each n, find a number $f(n)$, prove that that we always have $m \leq f(n)$, and show an example where $m = f(n)$.)*

Remark. It is an easy guess that $f(n) = \binom{n}{\lfloor n/2 \rfloor}$. This can be achieved by picking all $a_i = 1$ and A_i's being all the $\binom{[n]}{\lfloor n/2 \rfloor}$. We prove this is indeed the upper bound for m.
If we have $a_i \geq 1$ instead of $|a_i| \geq 1$, the problem is quite easy. The A_i's must form an antichain. For any $A \subset B$ will result in $S(B) \geq S(A) + 1$. Things get complicated when there are negative numbers.
As we said a couple of times, it makes a big difference how do we look at the objects: whether you look at them in the usual way, or upside down, or some of them in the usual way, and others upside down. A wonderful viewpoint is often the key. (Is this why Einstein is a great physician?)

Proof. Let $P := \{x : a_x \geq 1\}$ and $Q := \{x : a_x \leq -1\}$ be the partition of $[n]$. For any A_i, we transform it to

$$B_i = (A_i \cap P) \cup (Q - A_i).$$

If $B_i \subset B_j$, then (check) $S(A_j) - S(A_i) \geq 1$. So all the B_i's must form an antichain. And $m \leq \binom{n}{\lfloor n/2 \rfloor}$ by Sperner's theorem. □

Problem 4. *Let A_1, A_2, ..., A_m be m distinct subsets of $[n]$ such that each $|A_i|$ is even and for any $i \neq j$, $|A_i \cap A_j|$ is odd. How big can m be? Prove your answer. (i.e., Prove the bound, and show examples where the bound is reached.)*

Lemma 1. *The matrix $J_n - I_n$ over the binary field has rank n if n is even, and $n - 1$ if n is odd.*

Proof. In the binary field, plus and minus are the same, so the determinant (by its $n!$ terms expansion) equals the number of derangements. It is 1 iff n is even. When n is odd, the rank is less than n, but the first $n - 1$ rows and columns has full rank, so its ranks is $n - 1$. □

Solution. For n odd, the set $\binom{[n]}{n-1}$ satisfies the requirement and has size n. For n even, the set $\binom{[n-1]}{n-2}$ satisfies the requirement and has size $n - 1$. We prove this is best possible.

As in class, take the incidence matrix M where $M_{i,j} = 1$ if A_i contains j. As matrices over the binary field, $MM^T = J_m - I_m$.
If n is odd, there cannot be $n+1$ such sets. Otherwise, the $(n+1) \times (n+1)$ matrix $MM^T = J - I$ has full rank according to the lemma. But M has only n columns. A contradiction.
If n is even, there cannot be n such sets. Otherwise, the $n \times n$ matrix $J - I$ has full rank, but the columns vectors of M add up to the all 0 vector, so M has rank less than n. $\square$
NOTE: The proof for the last case is simpler than what I originally thought. I added an all 1 row to M, and a column with just one 1 to M^T, then show their product has full rank.