

Due: 2011/12/14 before class

## Homework 11

**Problem 1.** Let  $n$  be a positive integer,  $r < n/2$ ,  $A = \binom{[n]}{r}$ , and  $B = \binom{[n]}{r+1}$ . Construct a bipartite graph  $G$  on  $A + B$ , where  $x \in A$  is connected to  $y \in B$  iff  $x \subseteq y$  (i.e. the set  $y$  is  $x$  plus one element). Prove that there is a perfect matching for  $|A|$  in  $G$ .

*Remark.* This is similar to our proof that any regular bipartite graph has a perfect matching.

*Proof.* We use Hall's theorem. All we need to show is that the graph satisfies Hall's condition. i.e. for any  $X \subseteq A$ ,  $|N(X)| \geq |X|$ .

Suppose  $X$  is a subset of  $\binom{[n]}{r}$  where  $|X| = a$  and  $|N(X)| = b$ . Suppose there are  $m$  edges between  $X$  and  $N(X)$ . For each element in  $X$ , it has  $n - r$  neighbours in  $B$ . So  $m = a(n - r)$  (\*1). On the other hand each element in  $B$  has  $(r + 1)$  neighbours in  $A$ , some of them may not be in  $X$ , so  $m \leq b(r + 1)$  (\*2). Since  $r < n/2$ , we have  $r + 1 \leq n - r$ . (\*1) and (\*2) gives  $a \leq b$ .  $\square$

**Problem 2.** Prove that, when  $n$  is odd, there are exactly two extremal cases for Sperner's theorem.

*Proof.* Let  $n = 2k + 1$ . Let  $A = \binom{[n]}{k}$  and  $B = \binom{[n]}{k+1}$ . The two extremal cases are: either take all  $A$ , or all  $B$ . By the proof of Sperner's theorem, we know the size of each subset must be  $k$  or  $k + 1$ . Now we prove that we can not have  $\binom{n}{k}$  subsets where both  $A$  and  $B$  have some contribution.

Prove by contradiction. Suppose there is  $\emptyset \neq X \subseteq A$  and  $\emptyset \neq Y \subseteq B$  form an antichain of size  $T = \binom{n}{k}$ . Consider the bipartite graph between  $A$  and  $B$  as in the previous problem. Let  $Y' := N(X)$ . Since  $X \cup Y$  form an antichain,  $Y \cap Y' = \emptyset$ . So  $|Y| + |Y'| \leq T$ . Because we assumed  $|Y| + |X| = T$ . We show a contradiction by proving  $|X| < |Y'|$ .

Problem 1 tells that there is a perfect matching for  $A$ , so clearly  $|X| \leq |Y'|$ . The proof also tells that, for the equality to hold,  $Y'$  has no other neighbours outside  $X$ . But this means, in the bipartite graph  $(A, B)$ ,  $X \cup Y'$  is disconnected from the rest of the world.

However,  $(A, B)$  is connected: From any subset  $S \in A$ , we keep adding and removing one vertex each time, and get any other set in  $A$  or  $B$ .  $\square$

**Problem 3.** Given  $n$  real numbers  $a_1, \dots, a_n$  such that each  $|a_i| \geq 1$ . For any subset  $A \subseteq [n]$ , define  $S(A) = \sum_{i \in A} a_i$ . If  $A_1, \dots, A_m$  are  $m$  distinct subsets of  $[n]$  such that  $|S(A_i) - S(A_j)| < 1$  for any  $i \neq j$ , how big can  $m$  be? Justify your answer. (i.e., For each  $n$ , find a number  $f(n)$ , prove that that we always have  $m \leq f(n)$ , and show an example where  $m = f(n)$ .)

*Remark.* It is an easy guess that  $f(n) = \binom{n}{\lfloor n/2 \rfloor}$ . This can be achieved by picking all  $a_i = 1$  and  $A_i$ 's being all the  $\binom{[n]}{\lfloor n/2 \rfloor}$ . We prove this is indeed the upper bound for  $m$ .

If we have  $a_i \geq 1$  instead of  $|a_i| \geq 1$ , the problem is quite easy. The  $A_i$ 's must form an antichain. For any  $A \subset B$  will result in  $S(B) \geq S(A) + 1$ . Things get complicated when there are negative numbers.

As we said a couple of times, it makes a big difference how do we look at the objects: whether you look at them in the usual way, or upside down, or some of them in the usual way, and others upside down. A wonderful viewpoint is often the key. (Is this why Einstein is a great physician?)

*Proof.* Let  $P := \{x : a_x \geq 1\}$  and  $Q := \{x : a_x \leq -1\}$  be the partition of  $[n]$ . For any  $A_i$ , we transform it to

$$B_i = (A_i \cap P) \cup (Q - A_i).$$

If  $B_i \subset B_j$ , then (check)  $S(A_j) - S(A_i) \geq 1$ . So all the  $B_i$ 's must form an antichain. And  $m \leq \binom{n}{\lfloor n/2 \rfloor}$  by Sperner's theorem.  $\square$

**Problem 4.** Let  $A_1, A_2, \dots, A_m$  be  $m$  distinct subsets of  $[n]$  such that each  $|A_i|$  is even and for any  $i \neq j$ ,  $|A_i \cap A_j|$  is odd. How big can  $m$  be? Prove your answer. (i.e., Prove the bound, and show examples where the bound is reached.)

**Lemma 1.** The matrix  $J_n - I_n$  over the binary field has rank  $n$  if  $n$  is even, and  $n - 1$  if  $n$  is odd.

*Proof.* In the binary field, plus and minus are the same, so the determinant (by its  $n!$  terms expansion) equals the number of derangements. It is 1 iff  $n$  is even. When  $n$  is odd, the rank is less than  $n$ , but the first  $n - 1$  rows and columns has full rank, so its ranks is  $n - 1$ .  $\square$

*Solution.* For  $n$  odd, the set  $\binom{[n]}{n-1}$  satisfies the requirement and has size  $n$ . For  $n$  even, the set  $\binom{[n-1]}{n-2}$  satisfies the requirement and has size  $n - 1$ . We prove this is best possible.

As in class, take the incidence matrix  $M$  where  $M_{i,j} = 1$  if  $A_i$  contains  $j$ . As matrices over the binary field,  $MM^T = J_m - I_m$ .

If  $n$  is odd, there cannot be  $n + 1$  such sets. Otherwise, the  $(n + 1) \times (n + 1)$  matrix  $MM^T = J - I$  has full rank according to the lemma. But  $M$  has only  $n$  columns. A contradiction.

If  $n$  is even, there cannot be  $n$  such sets. Otherwise, the  $n \times n$  matrix  $J - I$  has full rank, but the columns vectors of  $M$  add up to the all 0 vector, so  $M$  has rank less than  $n$ .  $\square$

NOTE: The proof for the last case is simpler than what I originally thought. I added an all 1 row to  $M$ , and a column with just one 1 to  $M^T$ , then show their product has full rank.