Combinatorics and Graph Theory, Final Exam
brief solutions and remarks

General remark. As I mentioned in class. For the problems asking for an answer instead of a proof, you get full credit as long as the answer is correct. Other details are just evidences for partial credits.

There are 150 points on the paper. The last problem (20 points) was there to keep the best students warm. I expect someone can solve it only if s/he completes all the other problems and has one hour left. So, any score above 100 will be considered as full mark, and scores above 80 are quite good. The first problems are basics from the textbook and notes. The latter ones are harder.
The highest score is 118. The high scores are: 100+: 4 students; $[90, 100)$: 1 student; $[80, 90)$: 7 students.

Problem 1. *(5 points) How many edges does the graph for n-dimensional cube Q^n have? (Recall that the cube has all subsets of $[n]$ as vertices; there is an edge between A and B iff they differ by exactly 1 element.)*

Solution. There are 2^n vertices, each has degree n. So the number of edges is $n2^{n-1}$. □

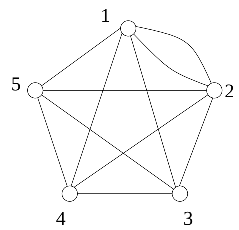
Problem 2. *(5 points) Show a colouring of the 7 points in the Fano configuration with 3 Y's, 3 R's, and 1 B, such that there is no monochromatic hyperedge.*

We omit the solution. There are many ways one can do this.

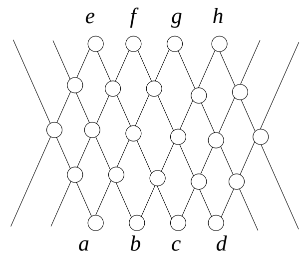
Problem 3. *(10 points)* (a). *Construct a graphs on* $[12]$ *with* 18 *edges, such that there is at least one edge among any 4 vertices.* (b). *Briefly justify that you cannot do it using* 17 *edges instead of* 18.

This is Turán's theorem in its complemented world. The extremal graph has 3 disjoint K_4, 18 egdes.

Problem 4. *(10 points) Let G be the graph in Figure (a) with parallel edges. How many spanning trees does it have?*



(a)



(b)

Solution. There are $5^3 = 125$ spanning trees in K_5. So there are $125 \cdot 4$ (tree, edge) pairs. For any edge (in our picture, $1 - 2$), there are $125 \cdot 4/10 = 50$ trees that contains that edge, and 75 that does not. Back to our problem, when $1 - 2$ is in the tree, we have two choices. So the number of spanning trees is $50 \cdot 2 + 75 = 175$. □

Note. See HW12, Prob 1.

Note. This can also be solved by the matrix-tree theorem. You get 8 points if you write down the correct 4×4 matrix.

Problem 5. *(10 points) As in Figure (b). 4 people start from a, b, c, d, respectively. They want to escape from the bottom line to the exits on the top. There are 4 exit points e, f, g, and h. Everyone needs to find a path with 4 steps along the lines in the picture (each step goes to a dot in the picture), and end up with one of the exit points. In addition, the paths of any two of them cannot share any common point (including the final exit). In how many ways can they do this?*

Solution. View this as a d.a.g. where all the edges going upwards, and all weights are 1. Use Lindström-Gessel-Viennot lemma, note that all the vertex disjoint paths must corresponding to the permutation $a \to e$, $b \to f$, $c \to g$, $d \to h$. The answer is the determinant of the matrix

$$((6,4,1,0),(4,6,4,1),(1,4,6,4),(0,1,4,6)).$$

The answer is 105. (When computing the determinant, you don't need to worry about the sign when doing row switch and negations, since you know the final answer is positive.) □

Note. See HW12, Prob 2.

Note. 8 points if you write the correct matrix and did not compute the determinant correctly.
Note. This can also solved by enumeration, which is slower, and not feasible if we have more rows.

Problem 6. *(10 points) Suppose $n > 1$ and $1 \le r \le n$. Let $\mathcal{A} \subseteq \binom{[n]}{r}$ be a collection of r-sets in $[n]$. Define its* shadow

$$\mathcal{B} = \left\{ B \in \binom{[n]}{r-1} : \exists A \in \mathcal{A}, B \subset A \right\}.$$

Prove the following inequality about their proportion in their own levels:

$$|\mathcal{A}|/\binom{n}{r} \le |\mathcal{B}|/\binom{n}{r-1}.$$

View this as a bipartite graph, count the number of edges between them.
Note. See HW11, Prob 1. We also did similar countings in the notes.
Note. We can also show the equality holds iff $\mathcal{A}$ is empty or the whole level. See HW11, Prob 2.
Note. This leeds to another proof for Sperner's theorem.

Problem 7. *(10 points) Suppose A is a set of 2011 integers, and for any $X \in \binom{A}{201}$, one can find $a, b \in X$ such that $a|b$. Prove that there is a set $Y \in \binom{A}{11}$ such that $a|b$ or $b|a$ for any $a, b \in Y$.*

Proof. Draw a directed graph on A where $a \to b$ if $a|b$. Clearly this is a d.a.g.. The problem states that the maximum antichain is at most 200. So, by Dilworth theorem, there are ≤ 200 chains whose union covers all A. By pigeonhole principle, there is one chain of size at least 11. $\square$

Problem 8. *(10 points) n people enter an elevator on the 1st floor. Each person randomly and uniformly selects one of the F upper floors, and pushes the corresponding button to request a stop at that floor. What is the expected number of stops the elevator will make on its way up? (The last stop also counts as one. The elevator stops at a floor if and only if someone requested it. And let's say $n \le 33$ for safety reasons.)*

Solution. Let X_i be the indicator r.v. that the elevator will stop at level i, i.e., there is at least one person requested level i. Let $X := \sum X_i$ be the number of stops. We have $\mathsf{E}(X_i) = 1 - (1 - 1/F)^n$. By linearity of expecatation,

$$\mathsf{E}(X) = F(1 - (1 - 1/F)^n).$$

□

Note. It is fine if you express the answer as a reasonable, correct expression, even if it is not in closed form.
Note. The condition $n \leq 33$ was useless. It was there just because such an elevator would be big enough to hold this class plus the instructor.

Problem 9. *(15 points) Let G be a graph whose chromatic number $\chi(G) = k$. Prove that G has at least k vertices with degrees no less than $k-1$.*

Proof. Let f be a colouring of the graph with k colours, we claim that, for any colour $c \in [k]$, there must be one vertex with degree $\geq k-1$ and coloured with c. Otherwise, w.l.o.g., suppose k does not have this property, let $S = \{x : f(x) = k\}$. We find a new colouring f', where $f'(y) = f(y)$ if $y \notin S$. For $x \in S$, $f'(x)$ is any colour in $[k-1]$ that was not used by the neighbours of x. (Since $\deg(x) < k-1$, this is always doable.)And it is easy to check (do it) f' is a proper colouring with only $k-1$ colours. A contradiction. □

Note. See HW7, Prob 2(d).
Note. In the exam, many students found alternative proofs that are quite simple. Here is one: Suppose there are at most $k-1$ such vertices, we colour them with $[k-1]$ in any manner, then colour the remaining vertices one by one, and we always have a good choice in each step.

Problem 10. *(15 points) Suppose we have n countries on a sphere of a planet that resembles a perfect potato. Each country has a connected interior region. Two countries are* adjacent *if they share some positive length of common border. Now they want to choose the national flags. Two adjacent countries cannot have the same flag. (This is the era before the air-travel, they don't care the non-adjacent fellows.)*
We know in each country, the citizens voted 4 candidates for their flag. However, due to the lack of imagination or the prevalence of burglaries, many of the designs look the same. Prove we can still pick the flags for at least $2n/3$ countries so that any two adjacent ones among them have different flags.

Proof. By the four color theorem, the countries can be partitioned into 4 parts, V_1, V_2, V_3, V_4, where there are no adjacent counties inside each part. Now uniformly randomly label each design with 1, 2, 3, or 4. For each country in V_i, we call it good if there is a flag design labeled as i, and we pick that design as its flag. In this way, there are no conflicts among the good countries. Let X_c be the indicator r.v. that c is a good country. $\mathsf{E}(X_c) = 1-(3/4)^4 > 2/3$. So the expected number of good countries is more than $2n/3$... ☐

Note. See HW6, Prob 5.

Problem 11. *(15 points) Let $k \geq 10$ and $n = \lfloor 2^k/(10k) \rfloor$. A subset $X \subseteq [n]$ is called a* Za *if the elements of X form an arithmetic progression of length k.*
(a). *Show that for any Za, there are less than $1.25kn$ other Za's share some common points with it.*
(b). *Prove that we can color each element of $[n]$ with yellow and blue such that no Za is monochromatic.*

Proof. (a). We prove any $x \in [n]$ is contained in less than $1.25n$ Za's. This is because a Za passing through x is decided by a pair (d, t), where d is the common difference of the arithmetic progression, and t is the rank of x in that progression. Clearly $1 \leq t \leq k$. The length of the interval $[1, n]$ is $n-1$, and the length of the Za is $d(k-1)$, so $d \leq (n-1)/(k-1)$. Therefore, the number of such Za's is at most $(n-1)k/(k-1) < 1.25n$ when $k \geq 10$.
(b). Randomly uniformly colour each vertex with Y or B. For each Za z, define E_z to be the event that z is monochromatic. So $\mathsf{Pr}(E_z) = 2^{1-k}$. Define a graph G on the events so that $E_z \to E_{z'}$ iff z and z' share some common points. Use (a) and L.L.L.. Alternatively, apply Example 8.4 in the notes. ☐

Problem 12. *(15 points) Little Moira is attending a party with 100 people (including herself). Two people are either friends to each other, or not.*
(a). *Prove that, if Moira has an odd number of friends in the party, then there exists one person who shares an even number of common friends with her.*
(b). *Prove that, no matter what is the friendship relation among the 100 people, there are two people who share an even number of common friends.*

Proof. (a). Let N be the neighbours of Moira. Look at the induced subgraoh $G' = G[N]$. is odd. Since $|N|$ is odd, there must be one point $x \in N$ such that $\deg_{G'}(x)$ is even. This happens iff x and Moira have an even number of friends.
(b). If there is a vertex with odd degree, we are done by (a). Otherwise, suppose for a contradiction we have all the degrees even, but $|N(u) \cap N(v)|$

odd for all $u \neq v$. Let A be the adjacency matrix for G over the binary field. On one hand, all the columns sum to 0, so $r(A) < n$. On the other hand, $AA^T = J - I$ has full rank when $n = 100$ is even. A contradiction. □

Note. My friend Moira is the only female TopCoder finalist in history. When I showed her this problem, she came up with a proof that does not use linear algebra. (Several students also found similar proofs in the exam.)

Proof. (from Moira herself) (a) stays the same. For (b), just consider all the edges between $N(x)$ and $V - N - \{x\}$. Counting the edges from the $V - N - \{x\}$ side will give us an odd number, while from the $N(x)$ side an even number. (You can easily fill in the details.) □

Problem 13. *(20 points) Suppose G is a graph on $[n]$ with possible parallel edges. And there are k edge disjoint spanning trees in G: T_1, T_2, ..., T_k such that any edge belongs to at most one T_i. Prove that G has at least k^{n-1} different spanning trees.*

Proof. We may throw away additional edges, only keep the k trees, and colour the edge of T_i with colour i. Thus we have a coloured graph with k colours, with $n - 1$ edges of each colour.
Prove by induction on k. The base case $k = 1$ is trivial. Let E_k be the set of $n - 1$ edges of T_k. We claim that for any $E' \in \binom{E_k}{s}$, there are $(k-1)^{n-1-s}$ at least spanning trees T such that $E(T) \cap E_k$ is exactly E'. Therefore the number of distinct spanning trees is at least

$$\sum_{i=0}^{n-1} \binom{n-1}{s} (k-1)^{n-1-s} = (1 + (k-1))^{n-1} = k^{n-1}.$$

To prove the claim, consider the "reduced" graph R where each vertex is a connected component in (V, E'), and draw an edge with colour $i < k$ between two components V_1 and V_2 if there was any edge of colour i between them in G. Now check that the reduced graph has spanning trees for each colour $i < k$, so, by induction, it has $(k-1)^{n-1-s}$ different spanning trees. Each of these trees corresponds to $n - 1 - s$ edges in the original graph, to which we can add back E' and form a spanning tree of the original graph. □

Note. As far as I remember, this problem (or at least the $k = 2$ version) was created by Józef Beck's in the classroom when he was teaching graph theory. I was awarded some dollars for first finding a proof (basically like above).

Think about that now, it was my bad that I forgot to offer money for nice solutions in the semester.